Dual-Series Solution to Scattering from a Semicircular Channel in a Ground Plane

M. K. Hinders and A. D. Yaghjian

Abstract—**Exact dual-series eigenfunction solutions, and simple closed-form low-frequency asymptotic approximations are determined for the problems of TM and TE scattering from a semicircular channel in a perfectly conducting ground plane. The eigenfunction solutions provide benchmarks for channel scattering, and the low-frequency solutions can be used to determine directly incremental length diffraction coefficients for narrow channels.**

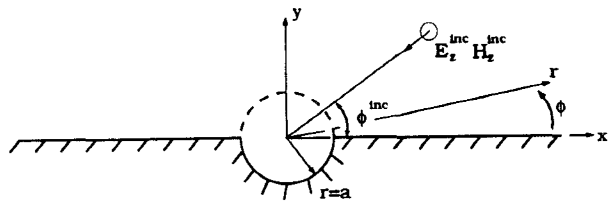

Fig. 1. Geometry of semicircular channel in perfectly conducting ground plane.

ALTHOUGH the scattering of plane waves from semicircular protuberances [1] and slits [2] have long been solved exactly, neither an exact eigenfunction solution nor a closed-form low-frequency solution exists, as far as we are aware, to the problem of scattering from a channel in a ground plane. We thus present in this letter dual-series eigenfunction solutions, and their low-frequency asymptotic approximations in closed form, to the 2-D problems of TM and TE scattering by a semicircular channel in a ground plane [3]. (A series representation for the low-frequency TM solution was found previously by Sachdeva and Hurd [4].) As shown in Fig. 1, a normally incident plane wave makes an angle ϕ^{inc} with the positive x axis, the channel along the z axis is of radius $r = a$, and the direction of scattering is given by the angle ϕ with the x axis. Harmonic time dependence of the form $e^{i\omega t}$ is assumed throughout.

For the incident TM plane wave, the z-component of the electric field can be expanded in cylindrical waves as [5]

$$E_z^{\text{inc}} = e^{ikr\cos(\phi-\phi^{\text{inc}})} = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in(\phi-\phi^{\text{inc}})} \quad (1)$$

The scattered field for $r \geq a$ may be expressed as the sum of two parts, the reflected TM plane wave given by

$$E_z^{\text{ref}} = -e^{ikr\cos(\phi+\phi^{\text{inc}})} = -\sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in(\phi+\phi^{\text{inc}})} \quad (2)$$

and the "diffracted" field expanded with Hankel functions as

$$E_z^{\text{dif}} = \sum_{n=1}^{\infty} A_n H_n^{(2)}(kr)\sin(n\phi) \qquad (r \geq a), \quad (3)$$

where the A_n are the unknown modal coefficients. The diffracted field vanishes on the ground plane so that the total field for $r \geq a$ also vanishes there.

In the interior region ($r \leq a$) the electric field can be expanded with Bessel functions as

$$E_z^{\text{int}} = \sum_{n=0}^{\infty} J_n(kr)\left[B_n\cos(n\phi) - C_n\sin(n\phi)\right] \qquad (C_0 = 0) \qquad (r \leq a), \quad (4)$$

where B_n and C_n are two more sets of modal coefficients which, along with A_n, will be determined from the boundary conditions at $r = a$. From Maxwell's equations we write the ϕ-component of the magnetic field as $H_\phi = (1/i\omega\mu)\partial_r E_z$, so that the boundary conditions at $r = a$ of zero tangential electric field on the channel ($\pi < \phi < 2\pi$) and continuous fields across the aperture circular ($0 < \phi < \pi$) become

$$\sum_{n=0}^{\infty} B_n J_n(ka)\cos(n\phi) + \sum_{n=1}^{\infty} C_n J_n(ka)\sin(n\phi) = 0 \qquad (\pi < \phi < 2\pi),$$

$$\sum_{n=0}^{\infty} B_n J_n(ka)\cos(n\phi) + \sum_{n=1}^{\infty} C_n J_n(ka)\sin(n\phi) = \sum_{n=1}^{\infty}\left[4i^n J_n(ka)\sin(n\phi^{\text{inc}}) + A_n H_n^{(2)}(ka)\right]\sin(n\phi) \qquad (0 < \phi < \pi),$$

$$\sum_{n=0}^{\infty} B_n J_n'(ka)\cos(n\phi) + \sum_{n=1}^{\infty} C_n J_n'(ka)\sin(n\phi) = \sum_{n=1}^{\infty}\left[4i^n J_n'(ka)\sin(n\phi^{\text{inc}}) + A_n H_n^{(2)\prime}(ka)\right]\sin(n\phi) \qquad (0 < \phi < \pi). \quad (5)$$

By making a simple change of variables $\phi \to \phi - \pi$ in the first equation, each equation in (5) can be written in the form

$$\sum_{n=0}^{\infty} g_n(ka)\cos(n\phi) = \sum_{n=1}^{\infty} f_n(ka)\sin(n\phi) \qquad (0 < \phi < \pi). \quad (6)$$

Partial orthogonality of the sinusoids over 0 to π gives the

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following relations between $f_m(ka)$ and $g_n(ka)$

$$f_m(ka)|_{m=0,2,4} = \frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{mg_n(ka)}{m^2-n^2}$$

$$f_m(ka)|_{n=1,3,5} = \frac{4}{\pi}\sum_{n=0,2,4}^{\infty}\frac{mg_n(ka)}{m^2-n^2}. \quad (7)$$

These relations can then be used to derive the following two equations

$$\sum_{\substack{n=0\\(n-m\ \text{odd})}}^{\infty}\left(\frac{B_n}{m^2-n^2}\right)\Big\{J_n(ka)+i\frac{\pi}{2}ka \cdot [J_n(ka)H_m^{(2)\prime}(ka)J_m(ka) - J_n'(ka)H_m^{(2)}(ka)J_m(ka)]\Big\} = \frac{\pi}{m}i^m J_m(ka)\sin(m\phi^{\text{inc}}), \quad m=1,2,3\cdots,$$

$$A_l = \left[\frac{8l}{\pi}\sum_{\substack{n=0\\(n-l\ \text{odd})}}^{\infty}\left(\frac{B_nJ_n(ka)}{l^2-n^2}\right) - 4i^lJ_l(ka)\sin(l\phi^{\text{inc}})\right]\frac{1}{H_l^{(2)}(ka)}; \quad l=1,2,3\cdots. \quad (8)$$

The first can be solved numerically to give the B_n and these can be inserted in the second to give A_l. This constitutes an eigenfunction solution of our TM problem.

For narrow channels (small ka) we can employ the small-argument approximations for the cylinder functions to write (8) as

$$\sum_{\substack{n=0\\(n-m\ \text{odd})}}^{\infty}\left(\frac{b_n}{m^2-n^2}\right)\left[\frac{3m+n}{2}\right] = i\frac{\pi}{2}ka\sin\phi^{\text{inc}}\delta_{1m} \quad m=1,2,3\cdots,$$

$$A_l = 4il\frac{(ka/2)^l}{l!}\left[\pi i^l\frac{(ka/2)^l}{l!}\sin(l\phi^{\text{inc}}) - \sum_{\substack{n=0\\(n-l\ \text{odd})}}^{\infty}\left(\frac{2lb_n}{l^2-n^2}\right)\right] \quad l=1,2,3\cdots. \quad (9)$$

Solving (9) numerically, we then find the following closed-form expression for TM scattering from a small semicircular channel

$$E_z^{\text{dif}} \underset{r\to\infty}{\sim} 0.185\sqrt{2\pi}(ka)^2 e^{i3\pi/4}\frac{e^{-ikr}}{\sqrt{kr}}\sin\phi\sin\phi^{\text{inc}} \quad (ka\ \text{small}), \quad (10)$$

which has the same functional form as that for TM scattering from a narrow slit in a conducting plane [2], [6]. In the expression (10) we have found the constant 0.185 by a single numerical matrix inversion since (9) shows that for narrow channels the ka-dependence is removed from the matrix, and only the coefficient A_1 is needed to determine the diffracted fields.

The transverse electric case can be solved in a similar fashion by expressing z-components of the incident, reflected, and diffracted H-fields as

$$H_z^{\text{inc}} = e^{ikr\cos(\phi-\phi^{\text{inc}})} = \sum_{n=-\infty}^{\infty} i^nJ_n(kr)e^{in(\phi-\phi^{\text{inc}})}$$

$$H_z^{\text{ref}} = e^{ikr\cos(\phi+\phi^{\text{inc}})} = \sum_{n=-\infty}^{\infty} i^nJ_n(kr)e^{in(\phi+\phi^{\text{inc}})}$$

$$H_z^{\text{dif}} = \sum_{n=0}^{\infty} A_n'H_n^{(2)}(kr)\cos n\phi \quad (r\ge a), \quad (11)$$

and the magnetic field in the interior region as

$$H_z^{\text{int}} = \sum_{n=0}^{\infty} J_n(kr)[B_n'\cos(n\phi)+C_n'\sin(n\phi)], \quad (C_0'=0) \quad (r\le a), \quad (12)$$

where A_n', B_n' and C_n' are the unknown modal coefficients to be determined from the boundary conditions. Proceeding as in the TM case, we find

$$\sum_{\substack{n=1\\(n-m\ \text{odd})}}^{\infty}\left(\frac{nC_n'}{n^2-m^2}\right)\Big\{J_n'(ka)-i\frac{\pi}{2}ka \cdot [J_n'(ka)H_m^{(2)}(ka)J_m'(ka) - J_n(ka)H_m^{(2)\prime}(ka)J_m'(ka)]\Big\} = \pi i^mJ_m'(ka)\cos(m\phi^{\text{inc}}) \quad m=0,1,2\cdots,$$

$$A_0' = \left[\frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\left(\frac{nC_n'J_n(ka)}{n}\right) - 2J_0'(ka)\right]\frac{1}{H_0^{(2)\prime}(ka)}$$

$$A_l' = \left[\frac{8}{\pi}\sum_{\substack{n=1\\(n-l\ \text{odd})}}^{\infty}\left(\frac{C_n'J_n'(ka)}{n^2-l^2}\right) - 4i^lJ_l'(ka)\cos(l\phi^{\text{inc}})\right]\cdot\frac{1}{H_l^{(2)\prime}(ka)} \quad l=1,2,3\cdots. \quad (13)$$

The first of (13) allows us to solve for the C_n', which can then be substituted into the others to get A_l'.

For small ka (13) can be shown to reduce to

$$\sum_{\substack{n=1\\(n-m\ \text{odd})}}^{\infty}\frac{nc_n'}{n^2-m^2}\left(\frac{3n+m}{2}\right) = i\pi\cos\phi^{\text{inc}}\delta_{1m} \quad m=1,2,3,\cdots,$$

$$\sum_{n=1,3,5} c_n' = -2\pi(ka/2)^2\left(1-\tfrac{1}{2}(ka)^2\ln(ka)\right) \quad m=0,$$

$$A_0' = \frac{\pi}{2i}(ka)^2\left[1-\tfrac{1}{2}(ka)^2\ln(ka)\right],$$

$$A_l' = \pi i\frac{(ka/2)^l}{l!}\left[\frac{8}{\pi}\sum_{\substack{n=1\\(n-l\ \text{odd})}}^{\infty}\left(\frac{n^2c_n'}{n^2-l^2}\right) - 4li^l\frac{(ka/2)^l}{l!}\cos(l\phi^{\text{inc}})\right] \quad 1=1,2,3\cdots, \quad (14)$$

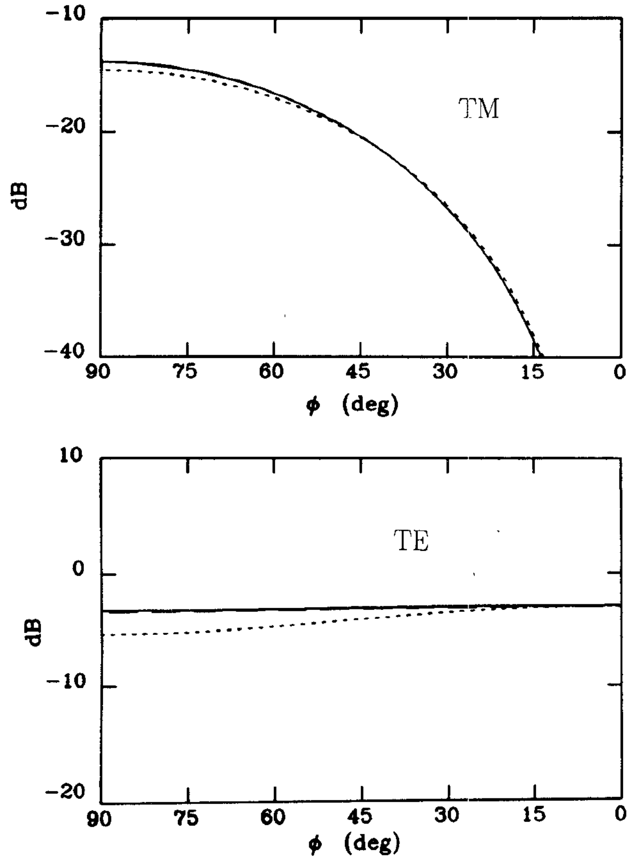

Fig. 2. Backscatter width/λ vs. scatter angle, $a/\lambda = 0.1$; —— eigenfunction, – – moment method, - - - low frequency.

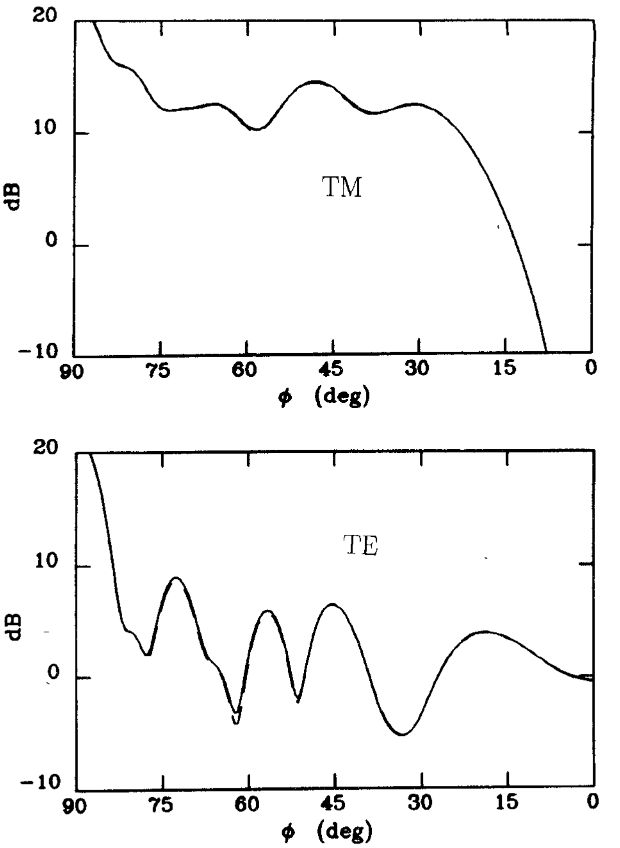

Fig. 3. Backscatter width/λ vs. scatter angle, $a/\lambda = 2$; —— eigenfunction, – – moment method.

from which is found the closed-form expression of the diffracted magnetic field for TE scattering from a narrow semicircular channel

$$H_z^{\text{dif}} \underset{r\to\infty}{\sim} \sqrt{2\pi}\,(ka)^2 e^{i3\pi/4}\,\frac{e^{-ikr}}{\sqrt{kr}} \cdot \left[-\tfrac{1}{2} + \frac{(ka)^2}{4}\ln(ka) - 0.185\cos\phi\cos\phi^{\text{inc}}\right]. \quad (15)$$

The leading term in (15) differs from the $1/\ln(ka)$ dependence of the leading term for TE scattering from a thin slit in a conducting plane [2], [6]. Note that we have once again found the numerical constant 0.185 via a single numerical matrix inversion, since (14) shows that the ka-dependence is removed from the matrix for narrow channels.

It has been shown recently [7] that the constants multiplying the TM magnetic dipole field in (10) and the TE electric dipole field in (15), are the same not just for the semicircular channel (where they are equal to 0.185) but for any cylindrical channel or ridge in a ground plane. Also the leading term in the TE low-frequency diffracted far magnetic field is always given by

$$\pm k^2\sqrt{2/\pi}\,\frac{e^{-i(kr+\pi/4)}}{\sqrt{kr}}\,A,$$

where A is the area of the channel or ridge [7]. Since the small-channel results show the k-dependence explicitly, they

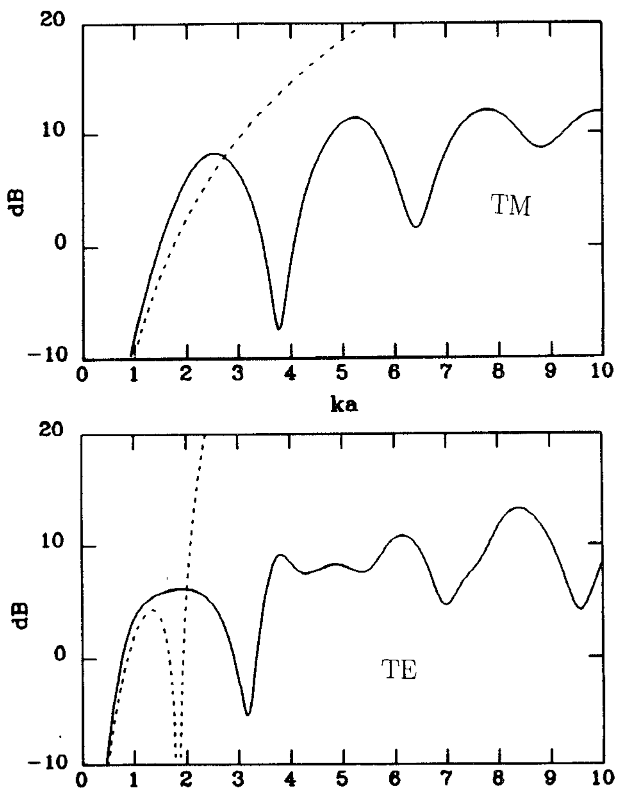

Fig. 4. Total scattering width/λ vs. ka, $\phi^{\text{inc}} = 90°$: —— eigenfunction, - - - low frequency.

can be generalized immediately to obliquely incident plane waves, and be used to determine incremental length diffraction coefficients [6].

Plots of the backscattering width versus angle computed from the TM and TE eigenfunction solutions, (8) and (13), as well as from a numerical moment-method integral equation solution [9], are shown in Figs. 2 and 3. The low-frequency approximations, (10) and (15) are also plotted in Fig. 2. The eigenfunction and moment-method solutions agreed well at all computed angles and frequencies. The low-frequency solutions are a good approximation to the exact eigenfunction solutions for a/λ smaller than about 0.1.

Fig. 4 shows the *total* scattering width versus ka in the TM and TE cases computed from the eigenfunction solutions and the low-frequency approximations. The clearly defined resonances displayed in Fig. 4 are not present in the total scattering width of the slit or the semicircular ridge on a ground plane.

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M. B. Woodworth programed the computation of the eigenfunction and low-frequency solutions and plotted the curves.

References

[1] J. W. Strutt (Lord Rayleigh), "On the light dispersed from fine lines ruled upon reflecting surfaces or transmitted by very narrow slits," *Phil. Mag.*, vol. 14, p. 350, 1907.

[2] J. J. Bowman, T. B. A. Senior, P. L. E. Uslenghi, *Electromagnetic and Acoustic Scattering by Simple Shapes.* Amsterdam: North-Holland, 1969, ch. 4 and section I.2.9.

[3] M. K. Hinders, "Scattering of a plane electromagnetic wave from a semicurcular crack in a perfectly conducting ground plane," RADC Tech. Rep., TR-89-12, Apr. 1989.

[4] B. K. Sachdeva and R. A. Hurd, "Scattering by a dielectric-loaded trough in a conducting plane," *J. Appl. Phys.*, vol. 48, no. 4, p. 1473, 1977.

[5] J. A. Stratton, *Electromagnetic Theory.* New York: McGraw-Hill, 1941, ch. 6.

[6] R. A. Shore and A. D. Yagnjian, "Incremental diffraction coefficients for planar surfaces," *IEEE Trans. Antennas Propagat.*, vol. 36, no. 1, p. 55, Jan. 1988.

[7] T. B. Hansen and A. D. Yaghjian, "Low-frequency scattering from two-dimensional perfect conductors," *IEEE/AP-S Symp. Dig.*, London, ON, Canada, June 1991; also Rome Laboratory Tech. Rept, TR-91-49, Jan. 1991.

[8] ——, "Incremental diffraction coefficients for cylinders of arbitrary cross section: Application to diffraction from ridges and channels in perfectly conducting surfaces," in *IEEE/AP-S Symp. Dig.*, London, ON, Canada, June 1991, pp. 794.

[9] H. Ling, private communication, Dept. Elect. and Comput. Eng., Univ. of Texas at Austin, 1990.